JOURNAL OF COMPUTATIONAL PHYSICS 104, 211–220 (1993)

An Accurate Solution of the Poisson Equation by the Chebyshev Collocation Method

H. DANG-VU

Institut de Mathématiques pures et appliquées, Université Pierre et Marie Curie, Paris, France

AND

C. DELCARTE

Equipe RAMSES, U. P. R. 293 du CNRS, Université Paris-Sud, Orsay, France

Received March 13, 1991; revised October 22, 1991

A new Chebyshev collocation method is presented for the 2D Poisson equation. The resolution of the mixed collocation τ equations leads to two quasi-tridiagonal systems which can be solved by standard techniques. Comparison of the results for the test problem $u(x, y) = \sin 4\pi x \sin 4\pi y$ (*J. Comput. Phys.* **30**, 167 (1979)), with those computed by Haidvogel and Zang, using the matrix diagonalization method, indicates that our method would be more accurate at large N values. © 1993 Academic Press, Inc.

1. INTRODUCTION

In Ref. [1] Haidvogel and Zang developed a matrix diagonalization method for the solution of the two-dimensional Poisson equation. This method is efficient but requires a preprocessing calculation of eigenvalues and eigenvectors which, as already mentioned by [1, 2], limits the accuracy of the solution to that of the preprocessing calculations, especially at large N values. In this paper we present a new method of solving the Poisson equation. This alternative method leads to the resolution of two quasi-tridiagonal systems and offers an accurate solution at large N.

Section 2 describes the algorithm for the 1D Poisson equation. The exact solution to the collocation equations is given. The result obtained in this section will be used in the next section. Section 3 is concerned with the 2D Poisson equation. We show that the Chebyshev-τ collocation equations are equivalent to two quasi-tridiagonal systems. Standard algorithms for solving tridiagonal systems can be modified to solve these systems. A direct solver based on an inverse matrix calculation is established. In Section 4 four examples have been included to demonstrate the application of the method. We point out that, in the test problem $u(x, y) = \sin 4\pi x \sin 4\pi y$, the results obtained by our method may be more accurate than those of Haidvogel and Zang when $N \geqslant 40$.

2. THE 1D POISSON EQUATION

Since the solution of the one-dimensional Poisson equation will be used in the two-dimensional case, in this section we shall compute this solution in detail. Consider the one dimensional equation,

$$u''(x) = f(x), \qquad -1 \leqslant x \leqslant 1, \tag{2.1}$$

with boundary conditions,

$$u(-1) = 0, \qquad u(1) = 0. \tag{2.2}$$

The computation of the solution to the problem (2.1)–(2.2) can be accomplished in many ways. One of the most commonly used techniques is the spectral method. This broad category of methods is generally considered to include Galerkin, collocation (sometimes called pseudospectral method), and τ methods, the type being determined by the basis functions chosen. A collocation method, the subject of this paper, can be summarized as follows: Let $x_0, x_1, ..., x_N$ be $N+1$ points of $[-1, 1]$, with $x_0 = 1$, $x_N = -1$. These points are called the collocation points. A particularly convenient choice for the collocation points x_j is

$$x_j = \cos(j\pi/N), \qquad j = 0, 1, ..., N. \tag{2.3}$$

On the interval $[-1, 1]$, let $\{\phi_k(x)\}$ be a set of complete, orthogonal basis functions, which has inner product sym-

0021-9991/93 $5.00

581/104/1-15

bolized by $\langle \cdot, \cdot \rangle$. In the Chebyshev collocation method we choose

$$\phi_k(x) = T_n(x),$$

where $T_n(x)$ denotes the Chebyshev polynomial of degree n defined by $T_n(\cos\theta) = \cos(n\theta)$ when $x = \cos\theta$. The points x_j defined by (2.3) are then the extrema of $T_n(x)$.

The set of Chebyshev polynomials $T_n(x)$ is an orthogonal system with respect to the inner product defined by

$$\langle T_i, T_j \rangle = \int_{-1}^{1} T_i(x)\, T_j(x) \frac{dx}{\sqrt{1-x^2}}, \tag{2.4}$$

namely,

$$\begin{aligned} &\langle T_i, T_j \rangle = 0, && \text{if} \quad i \neq j \\ &\langle T_i, T_i \rangle = c_i \frac{\pi}{2}, && i = 0, 1, 2, ..., \end{aligned} \tag{2.5}$$

where $c_0 = 2$ and $c_n = 1$ for $n > 0$.

Each term in the differential equation (2.1), along with each term in boundary conditions (2.2), is expanded as a truncated series of Chebyshev polynomials as

$$u(x) \approx \sum_{n=0}^{N} a_n T_n(x)$$

$$f(x) \approx \sum_{n=0}^{N} f_n T_n(x)$$

$$u''(x) \approx \sum_{n=0}^{N} a_n^{(2)} T_n(x).$$

At this point it may be worthwhile for the reader to note that the basis functions do not satisfy the boundary conditions individually. (In the Galerkin method the basis functions must individually satisfy the boundary conditions. For example, a Chebyshev Galerkin approximation to the problem (2.1)–(2.2) would use the basis functions [4]: $\phi_n(x) = T_n(x) - T_0(x)$, for n even, and $\phi_n(x) = T_n(x) - T_1(x)$, for n odd).

Due to the orthogonality relation (2.5) we have the inversion formulas:

$$a_n = \frac{2}{N\bar{c}_n} \sum_{i=0}^{N} \frac{1}{\bar{c}_i} u(x_i)\, T_n(x_i)$$

$$f_n = \frac{2}{N\bar{c}_n} \sum_{i=0}^{N} \frac{1}{\bar{c}_i} f(x_i)\, T_n(x_i),$$

where $\bar{c}_0 = \bar{c}_N = 2$, $\bar{c}_i = 1$ for $0 < i < N$. For convenience we suppose that N is even and $N \geqslant 6$.

The coefficients of the second derivative are given by [4]

$$a_n^{(2)} = \frac{1}{c_n} \sum_{\substack{p=n+2 \\ p \equiv n (\mathrm{mod}\, 2)}}^{N} p(p^2 - n^2)\, a_p.$$

The notation $p \equiv n \pmod 2$ means that the sum includes $p = n+2$, $n+4$, etc.

Using these results, (2.1) gives the Chebyshev collocation equations:

$$\frac{1}{c_n} \sum_{\substack{p=n+2 \\ p \equiv n (\mathrm{mod}\, 2)}}^{N} p(p^2 - n^2)\, a_p = f_n,$$

$$0 \leqslant n \leqslant N-2. \tag{2.6}$$

It may be observed that a_0 and a_1 do not appear in (2.6). In fact they are given by (2.2) which can be written as

$$\sum_{\substack{p=0 \\ p \equiv 0 (\mathrm{mod}\, 2)}}^{N} a_p = 0, \qquad \sum_{\substack{p=1 \\ p \equiv 1 (\mathrm{mod}\, 2)}}^{N} a_p = 0. \tag{2.7}$$

An exact solution to (2.6) is obtained by using the recursion relation [3, 5],

$$c_{n-1} a_{n-1}^{(q)} - a_{n+1}^{(q)} = 2n a_n^{(q-1)}, \qquad n, q \geqslant 1,$$

where $a_n^{(q)}$ are the expansion coefficients of the derivative $u^{(q)}(x)$. Then it may be shown that the exact solution to (2.6) is

$$a_i = r_i f_{i-2} + s_i f_i + v_i f_{i+2}, \qquad 2 \leqslant i \leqslant N, \tag{2.8}$$

where

$$\begin{aligned} &r_i = \frac{c_{i-2}}{4i(i-1)}, \qquad s_i = -\frac{e_{i+2}}{2(i^2-1)}, \\ &v_i = \frac{e_{i+4}}{4i(i+1)}, \qquad 2 \leqslant i \leqslant N. \end{aligned} \tag{2.9}$$

Here $c_0 = 2$, $c_i = 1$ for $i > 0$ and $e_i = 1$ for $i \leqslant N$, $e_i = 0$ for $i > N$.

Finally, a_0 and a_1 are given by (2.7)

$$\begin{aligned} a_0 &= -(a_2 + a_4 + \cdots + a_N) \\ a_1 &= -(a_3 + a_5 + \cdots + a_{N-1}). \end{aligned} \tag{2.10}$$

Let P be the upper triangular $(N+1)\times(N+1)$ matrix

$$P=\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 \\ & 1 & 0 & 1 & 0 & 1 & \cdots & 0 \\ & & 2^3/2 & 0 & 4^3/2 & 0 & \cdots & N^3/2 \\ & & & 3(3^2-1^2) & 0 & 5(5^2-1^2) & \cdots & 0 \\ & & & & 4(4^2-2^2) & 0 & \cdots & N(N^2-2^2) \\ & & & & & 5(5^2-3^2) & \cdots & 0 \\ & & & & & & & \vdots \\ & & & & & & & N(N^2-(N-2)^2) \end{pmatrix} \tag{2.11}$$

and let a, f be column vectors of length $N+1$ defined by

$$a^{\mathrm{T}}=(a_0, a_1, a_2, \ldots, a_N)$$
$$f^{\mathrm{T}}=(0, 0, f_0, f_1, \ldots, f_{N-2});$$

then (2.6) in combination with (2.7) can be written as

$$Pa=f \tag{2.12}$$

and (2.8)–(2.10) is the solution to (2.12).

Let $Q=P^{-1}$, then Q is an upper triangular matrix of the form

$$Q=\begin{pmatrix} 1 & 0 & q_{02} & 0 & q_{04} & 0 & q_{06} & \cdots & 0 & q_{0N} \\ & 1 & 0 & q_{13} & 0 & q_{15} & 0 & \cdots & q_{0,N-1} & 0 \\ & & q_{22} & 0 & q_{24} & 0 & q_{26} & & & \\ & & & \cdots & \cdots & \cdots & & & & \\ & & & & q_{ii} & 0 & q_{i,i+2} & 0 & q_{i,i+4} & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & & q_{N-1,N-1} & 0 \\ & & & & & & & & & q_{N,N} \end{pmatrix}. \tag{2.13}$$

The nonzero entries q_{ij} are given by

$$q_{ii}=r_i, \qquad q_{i,i+2}=s_i,$$
$$q_{i,i+4}=v_i, \qquad 2\leqslant i\leqslant N,$$
$$q_{ij}=-\sum_{\substack{k=i+2\\ k\equiv i(\mathrm{mod}\,2)}}^{j} q_{kj},$$
$$i=0, 1; \quad j=i+2, i+4, \ldots.$$

These formulas will be used in the next section.

Finally it is worth noting that the eigenvalues of the matrix P are $1, 1, 2^3/2, 3(3^2-1^2), 4(4^2-2^2), (5(5^2-3^2), \ldots, N(N^2-(N-2)^2)=4N(N-1)$, whereas the matrix A of Haidvogel and Zang [1], representing the Chebyshev approximation to $\partial^2/\partial x^2$ with boundary conditions (2.7) has the largest eigenvalue $|\lambda_{\max}(A)|\sim O(N^4)$ [2]. In fact, let $\bar{A}$ be the $(N+1)\times(N+1)$ matrix,

$$\bar{A}=\begin{pmatrix} & & & 0 & 0 \\ & A & & \vdots & \vdots \\ & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

where A is the $(N-1)\times(N-1)$ matrix defined in [1]. Then we have the relation

$$\bar{A}=HP, \tag{2.14}$$

where H denotes the $(N+1)\times(N+1)$ matrix:

$$\begin{pmatrix} -N^3/2 & 0 & 1 & & & & & \\ 0 & -N(N^2-1^2) & 0 & 1 & & & & \\ -N(N^2-2^2) & 0 & 0 & 0 & 1 & & & \\ \cdots & \cdots & \cdots & & & & & \\ 0 & -N(N^2-(N-3)^2) & 0 & & \cdots & 0 & 1 & \\ -N(N^2-(N-2)^2) & 0 & 0 & & \cdots & & 0 & 1 \\ 0 & 0 & 0 & & \cdots & & & 0 \\ 0 & 0 & 0 & & \cdots & & & 0 \end{pmatrix}.$$

3. THE 2D POISSON EQUATION

In the Chebyshev pseudospectral approach to the two-dimensional Poisson equation in a square:

$$\Delta u\,(x, y) = f(x, y), \qquad x, y \in]-1, +1[\tag{3.1}$$

with homogeneous Dirichlet boundary conditions,

$$u(\pm 1, y) = u(x, \pm 1) = 0, \tag{3.2}$$

where u, f, and Δu are approximated by truncated double Chebyshev series,

$$u(x, y) \approx \sum_{n=0}^{N} \sum_{m=0}^{N} a_{nm} T_n(x)\, T_m(y)$$

$$f(x, y) \approx \sum_{n=0}^{N} \sum_{m=0}^{N} f_{nm} T_n(x)\, T_m(y) \tag{3.3}$$

$$\Delta u\,(x, y) \approx \sum_{n=0}^{N} \sum_{m=0}^{N} a_{nm}^{(2)} T_n(x)\, T_m(y).$$

The expansion coefficients a_{nm}, f_{nm}, $a_{nm}^{(2)}$ obtained using the collocation points $x_j = \cos(j\pi/N)$, $j = 0, 1, ..., N$, are given by [3]

$$a_{nm} = \frac{4}{N^2 \bar{c}_n \bar{c}_m} \sum_{i=0}^{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_i \bar{c}_j} \times a(x_i, y_i)\, T_n(x_i)\, T_m(x_j)$$

$$f_{nm} = \frac{4}{N^2 \bar{c}_n \bar{c}_m} \sum_{i=0}^{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_i \bar{c}_j} \times f(x_i, y_j)\, T_n(x_i)\, T_m(x_j)$$

$$a_{nm}^{(2)} = \frac{1}{c_n} \sum_{\substack{p=n+2 \\ p \equiv n (\mathrm{mod}\, 2)}}^{N} p(p^2 - n^2)\, a_{pm} + \frac{1}{c_m} \sum_{\substack{q=m+2 \\ q \equiv m (\mathrm{mod}\, 2)}}^{N} q(q^2 - m^2)\, a_{nq}.$$

For convenience we suppose that N is even and $N \geqslant 6$. The equations for a_{nm} that follow from (3.1)–(3.2) are then

$$\frac{1}{c_n} \sum_{\substack{p=n+2 \\ p \equiv n (\mathrm{mod}\, 2)}}^{N} p(p^2 - n^2)\, a_{pm} + \frac{1}{c_m} \sum_{\substack{q=m+2 \\ q \equiv m (\mathrm{mod}\, 2)}}^{N} q(q^2 - m^2)\, a_{nq} = f_{nm} \tag{3.4}$$

for $0 \leqslant n, m \leqslant N-2$,

$$\sum_{\substack{p=0 \\ p \equiv 0 (\mathrm{mod}\, 2)}}^{N} a_{pm} = 0, \qquad \sum_{\substack{p=1 \\ p \equiv 1 (\mathrm{mod}\, 2)}}^{N} a_{pm} = 0, \qquad 0 \leqslant m \leqslant N \tag{3.5}$$

$$\sum_{\substack{q=0 \\ q \equiv 0 (\mathrm{mod}\, 2)}}^{N} a_{nq} = 0, \qquad \sum_{\substack{q=1 \\ q \equiv 1 (\mathrm{mod}\, 2)}}^{N} a_{nq} = 0, \qquad 0 \leqslant n \leqslant N. \tag{3.6}$$

Equations (3.5)–(3.6) give only $4N$ independent equations since the four corners of the square have been counted twice. In effect four of the conditions (3.5), corresponding to $m = N-1$, N, for example, are linearly dependent upon the others. Consider, for example, the first equation of (3.5) with $m = N$,

$$\sum_{\substack{p=0 \\ p \equiv 0 (\mathrm{mod}\, 2)}}^{N} a_{pN} = 0.$$

Due to (3.6), the left-hand side of the above equation can be written as

$$\sum_{\substack{p=0 \\ p \equiv 0 (\mathrm{mod}\, 2)}}^{N} a_{pN} = \sum_{\substack{p=0 \\ p \equiv 0 (\mathrm{mod}\, 2)}}^{N} \left(- \sum_{\substack{q=0 \\ q \equiv 0 (\mathrm{mod}\, 2)}}^{N-2} a_{pq} \right) = - \sum_{\substack{q=0 \\ q \equiv 0 (\mathrm{mod}\, 2)}}^{N-2} \left(\sum_{\substack{p=0 \\ p \equiv 0 (\mathrm{mod}\, 2)}}^{N} a_{pq} \right) = 0.$$

Thus (3.4)–(3.6) gives $(N+1) \times (N+1)$ equations for the $(N+1) \times (N+1)$ unknowns a_{nm} $(0 \leqslant n \leqslant N,\ 0 \leqslant m \leqslant N)$. Haidvogel and Zang [1] stated, and we concur, that this approach of dropping the equations for the highest modes from (3.4) and imposing the $4N$ conditions (3.5)–(3.6) as part of the linear system determining the a_{nm}, $0 \leqslant n, m \leqslant N$, amounts to Lanczos' τ method [6].

Solution to the problem (3.4)–(3.6) can be obtained by calculations very similar to those for (2.6)–(2.7). If we define the column vectors A_i, F_i by

$$A_i^{\mathrm{T}} = (a_{i0}, a_{i1}, ..., a_{iN}), \qquad 0 \leqslant i \leqslant N$$

$$F_i^{\mathrm{T}} = (0, 0, f_{i0}, f_{i1}, ..., f_{i,N-2}), \qquad 0 \leqslant i \leqslant N$$

and let $\tilde{I}$ be the $(N+1)\times(N+1)$ matrix,

$$\tilde{I}=\begin{pmatrix} 0 & & & & \\ 0 & & & & \\ 1 & & \ddots & & \\ & \ddots & & & \\ & & 1 & 0 & 0 \end{pmatrix}, \qquad (3.7)$$

then the systems (3.4)–(3.6) can be written as

$$P\otimes\tilde{I}\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}+\tilde{I}\otimes P\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ F_0 \\ \vdots \\ F_{N-2} \end{pmatrix}, \qquad (3.8)$$

where $\tilde{I}\otimes P$ and $P\otimes\tilde{I}$ are tensor products defined by

$$\tilde{I}\otimes P=\begin{pmatrix} 0 & & & & \\ 0 & & & & \\ P & & \ddots & & \\ & \ddots & & & \\ & & P & 0 & 0 \end{pmatrix}$$

and

$$P\otimes\tilde{I}=\begin{pmatrix} \tilde{I} & 0 & \tilde{I} & 0 & \tilde{I} & 0 & \cdots & \tilde{I} \\ & \tilde{I} & 0 & \tilde{I} & 0 & \tilde{I} & \cdots & 0 \\ & & (2^3/2)\tilde{I} & 0 & (4^3/2)\tilde{I} & 0 & \cdots & (N^3/2)\tilde{I} \\ & & & 3(3^2-1^2)\tilde{I} & 0 & 5(5^2-1^2)\tilde{I} & \cdots & 0 \\ & & & & 4(4^2-2^2)\tilde{I} & 0 & \cdots & N(N^2-2^2)\tilde{I} \\ & & & & & 5(5^2-3^2)\tilde{I} & \cdots & 0 \\ & & & & & & & \vdots \\ & & & & & & & N(N^2-(N-2)^2)\tilde{I} \end{pmatrix},$$

where P is the $(N+1)\times(N+1)$ matrix defined by (2.11). If we put

$$\hat{P}=P\otimes I, \qquad \hat{a}=\begin{pmatrix} \tilde{I}A_0 \\ \tilde{I}A_1 \\ \tilde{I}A_2 \\ \vdots \\ \tilde{I}A_N \end{pmatrix},$$

$$\hat{g}=\begin{pmatrix} 0 \\ 0 \\ F_0-PA_0 \\ \vdots \\ F_{N-2}-PA_{N-2} \end{pmatrix},$$

where I is the identity matrix of order $(N+1)$, then (3.8) becomes

$$\hat{P}\hat{a}=\hat{g}. \qquad (3.9)$$

The system (3.9) has the same structure as the system (2.12) so that we can write (3.9) as

$$\tilde{I}A_i=r_i(F_{i-2}-PA_{i-2})+s_i(F_i-PA_i)+v_i(F_{i+2}-PA_{i+2}), \quad 2\leqslant i\leqslant N, \qquad (3.10)$$

$$A_N=-(A_0+A_2+\cdots+A_{N-2}) \qquad (3.11\text{a})$$

$$A_{N-1}=-(A_1+A_3+\cdots+A_{N-3}), \qquad (3.11\text{b})$$

where r_i, s_i, and v_i are given by (2.9).

The idea of the τ method [6] is to apply (3.10) for $i=2, 3, ..., N-2$ only. Equations for $i=N$ and $i=N-1$ are obtained by substituting (3.11a) and (3.11b), respectively, into the left-hand side of (3.10). In this way there result $N-1$ following equations:

$$\tilde{I}A_i=r_i(F_{i-2}-PA_{i-2})+s_i(F_i-PA_i)+v_i(F_{i+2}-PA_{i+2}), \quad 2\leqslant i\leqslant N-2, \qquad (3.12)$$

$$-\tilde{I}(A_0+A_2+\cdots+A_{N-2})=r_N(F_{N-2}-PA_{N-2}) \qquad (3.13\text{a})$$

$$-\tilde{I}(A_1+A_3+\cdots+A_{N-3})=r_{N-1}(F_{N-3}-PA_{N-3}). \qquad (3.13\text{b})$$

Note that the even and odd coefficients are uncoupled in (3.12) and (3.13). The structure of the linear system for the even coefficients is quasi-tridiagonal, namely

TABLE IV

Maximum Residue $\varepsilon = L_N u_N - f_N$ for Eq. (3.19) as a Function of N

N	ε
16	3.33×10^{-2}
24	2.52×10^{-5}
32	3.03×10^{-10}
40	2.55×10^{-14}

Note. The Helmholtz parameter k is 10^{-4}. The exact solution is $u(x, y) = \sin 4\pi x \sin 4\pi y$.

which is treated by [2]. For the lower N values, our results are comparable to that of [2]. For $N = 40$ our result is more accurate.

EXAMPLE 4. As the final test, we present applications of our method to the solution of the heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u(x, y, t) - f(x, y), \\ &\qquad x, y \in \left]-1, +1\right[\\ &u(x, y, 0) = u_0(x, y) \\ &u(\pm 1, y, t) = u(x, \pm 1, t) = 0. \end{aligned} \tag{4.4}$$

A finite difference scheme for temporal discretization can be used to solve this equation. The scheme reads as follows:

$$\frac{u^{n+1} - u^n}{\Delta t} = \theta(\Delta u)^{n+1} + (1 - \theta)(\Delta u)^n - f,$$

$$\frac{1}{2} \leqslant \theta \leqslant 1.$$

The case $\theta = 1$ corresponds to the backward Euler scheme and the case $\theta = \frac{1}{2}$ gives the Crank–Nicolson scheme. Therefore, at each time step, we have to solve a Helmholtz equation.

As a simple example, we assume that $f(x, y)$ has the form (4.2). As $t \to \infty$, the solution to (4.4) approaches the steady-state solution

$$u(x, y) = (x^2 - 1)(y^2 - 1)\, e^{x+y}, \qquad x, y \in \left]-1, +1\right[.$$

TABLE V

Errors in Steady-State Solution of (4.4)

N	Maximum relative error
10	8.26×10^{-5}
12	4.97×10^{-9}
16	3.66×10^{-13}

Note. The exact solution is $u(x, y) = (x^2 - 1)(y^2 - 1)\, e^{x+y}$.

In Table V, we list the relative maximum pointwise errors in the solution of this problem as $t \to \infty$. The calculations have been done with $\theta = \frac{1}{2}$ (Crank–Nicolson).

It should be noted that the results reported in Table V depend upon the parameter Δt and the number of iterations used. However, a convergence analysis for the time-discretization problems is beyond the scope of the present paper and can be found in [10].

5. CONCLUSION

A new collocation method for solution of the Poisson equation has been presented that is conceived to be of use in time-dependent problems. This is obtained by considering a direct method avoiding any iterative procedure. Computational results have been presented demonstrating the accuracy and reliability of the method. Compared with the Haidvogel–Zang method our method should generate more accurate results at large N values, but the Haidvogel–Zang algorithm seems faster than our algorithm.

So far we have concentrated on the domains which are a square or a rectangle. If the domain is the union of several rectangles (like an L-shape domain) then our calculations can be formulated in each of the rectangles. Across the internal boundaries, continuity of the solution and its normal derivative is required. An iterative scheme that reduces the problem to a sequence of mixed boundary value problems on each rectangle can be then devised: an arbitrary initial guess is made for the unknown solution (or its normal derivative) at the interface points. The iteration will proceed by the separate solution of the Poisson problem on each rectangle and the subsequent relaxation of the interface values for the next iteration. This procedure, proposed by Orszag [11], is known as the "patching-collocation method." A rigorous convergence analysis for the above procedure is given in Funaro *et al.* [12]. They also discuss the application of this iteration strategy to a two-dimensional Poisson problem and a two-dimensional Helmholtz problem with two subdomains. (The generalization to an arbitrary number of subdomains is straightforward.) Moreover, they indicate an effective strategy for the dynamical selection of the relaxation parameter. Their numerical results show that this iterative method is in general very fast. The extension outlined above is under investigation and results will be presented elsewhere.

APPENDIX

A. *Computation of the Direct Solution to* (3.15)

Let M be the block-matrix defined in (3.15). The entries $M_{2i,2j}$, $0 \leqslant i, j \leqslant \frac{1}{2}N-1$, of M are defined as

$$M_{00} = M_{02} = \cdots = M_{0,N-4} = \tilde{Q},$$
$$M_{0,N-2} = \tilde{Q} - r_N I$$
$$M_{N-2i,N-2i-2} = r_{N-2i} I,$$
$$M_{N-2i,N-2i} = \tilde{Q} + s_{N-2i} I,$$
$$M_{N-2i,N-2i+2} = v_{N-2i} I,$$
$$1 \leqslant i \leqslant \tfrac{1}{2}N-1,$$

where r_i, s_i, v_i are given by (2.9) and $\tilde{Q}$ is defined by (3.16).

Then the system (3.15) can be written as

$$\tilde{Q}\bar{A}_0 + \tilde{Q}\bar{A}_2 + \cdots + \tilde{Q}\bar{A}_{N-4} + M_{0,N-2}\bar{A}_{N-2} = -H_N \tag{A.1}$$

$$r_{N-2i}\bar{A}_{N-2i-2} + M_{N-2i,N-2i}\bar{A}_{N-2i} + v_{N-2i}\bar{A}_{N-2i+2} = H_{N-2i}, \quad 2 \leqslant i \leqslant \tfrac{1}{2}N-1 \tag{A.2}$$

$$r_{N-2}\bar{A}_{N-4} + M_{N-2,N-2}\bar{A}_{N-2} = H_{N-2}. \tag{A.3}$$

We seek recursion relations of the form:

$$\bar{A}_{N-2i+2} = \alpha_{2i-2}\bar{A}_{N-2i} + \beta_{2i-2}.$$

Substituting the above expression into (A.2) we obtain

$$\bar{A}_{N-2i} = \alpha_{2i}\bar{A}_{N-2i-2} + \beta_{2i} \tag{A.4}$$

$$\alpha_{2i} = -B_{2i}^{-1} r_{N-2i} \tag{A.5}$$

$$\beta_{2i} = B_{2i}^{-1}(H_{N-2i} - v_{N-2i}\beta_{2i-2}) \quad 1 \leqslant i \leqslant \tfrac{1}{2}N-1, \tag{A.6}$$

where

$$B_{2i} = M_{N-2i,N-2i} + v_{N-2i}\alpha_{2i-2}.$$

Equation (A.3) shows that

$$\alpha_0 = \beta_0 = 0. \tag{A.7}$$

Using the recursion relations (A.4), we can write (A.1) as

$$\tilde{Q}\bar{A}_0 = -\gamma_{N-4}\bar{A}_2 - H_N - (\gamma_{N-6}\beta_{N-4} + \cdots + \gamma_2\beta_4 + \gamma_0\beta_2). \tag{A.8}$$

Here γ_{2i} are defined by the recursion relations

$$\gamma_0 = M_{0,N-2}$$
$$\gamma_{2i} = \tilde{Q} + \gamma_{2i-2}\alpha_{2i}, \qquad 1 \leqslant i \leqslant \tfrac{1}{2}N-2. \tag{A.9}$$

We invoke the recursion relation (A.4) with $i = \frac{1}{2}N-1$:

$$\bar{A}_2 = \alpha_{N-2}\bar{A}_0 + \beta_{N-2}. \tag{A.10}$$

Equation (A.8) combined with (A.10) yields

$$\bar{A}_0 = -C_N^{-1}(H_N + \gamma_{N-4}\beta_{N-2} + \cdots + \gamma_2\beta_4 + \gamma_0\beta_2), \tag{A.11}$$

where

$$C_N = \tilde{Q} + \gamma_{N-4}\alpha_{N-2}. \tag{A.12}$$

$\bar{A}_2, \bar{A}_4, \ldots, \bar{A}_{N-2}$ are then given by (A.4). Thus the algorithm for the solution to (3.15) is

(1) Compute the matrices α_{2i} and β_{2i}, $1 \leqslant i \leqslant \frac{1}{2}N-1$, using (A.5) and (A.6), respectively.

(2) Compute the matrices γ_{2i}, $1 \leqslant i \leqslant \frac{1}{2}N-2$, using (A.9).

(3) Compute the column vector $\bar{A}_0$, using (A.11).

(4) Compute $\bar{A}_2, \bar{A}_4, \ldots, \bar{A}_{N-2}$, using the recurrence relations (A.4).

The matrices B_{2i}^{-1}, C_N^{-1} may be evaluated by standard techniques. In this paper they are computed by using the Gauss–Jordan method with diagonal pivots.

B. *Computation of* $(M^{-1})_{2i,2j}$

Equation (A.6) may be written

$$\beta_{2i} = \sum_{j=i}^{1} b_{N-2i,N-2j} H_{N-2j}, \quad 1 \leqslant i \leqslant \frac{N}{2}-1, \tag{A.13}$$

where $b_{2i,2j}$, $1 \leqslant i, j \leqslant N/2-1$, is an upper triangular bloc matrix whose entries are defined by

$$b_{N-2i,N-2i} = B_{2i}^{-1}, \qquad 1 \leqslant i \leqslant N/2-1,$$
$$b_{N-2i,N-2j+2} = -b_{N-2i,N-2j+2}v_{N-2j}B_{2j-2}^{-1}, \tag{A.14}$$
$$2 \leqslant i \leqslant N/2-1, j = i, i-1, \ldots, 2.$$

Substituting (A.13) into (A.11) we obtain

$$\bar{A}_0 = -(M^{-1})_{00} H_N + \sum_{j=1}^{N/2-1} (M^{-1})_{0,2j} H_{2j},$$

where

$$(M^{-1})_{00} = C_N^{-1}$$
$$(M^{-1})_{0,2j} = -C_N^{-1} \sum_{k=1}^{j} \gamma_{N-2k-2} b_{2k,2j}, \qquad \text{(A.15)}$$
$$1 \leqslant j \leqslant N/2-1.$$

One can then substitute (A.13) and (A.15) into (A.4) and obtain

$$(M^{-1})_{2i,0} = \alpha_{N-2i}(M^{-1})_{2i-2,0}$$
$$(M^{-1})_{2i,2j} = \alpha_{N-2i}(M^{-1})_{2i-2,2j} + b_{2i,2j}, \qquad \text{(A.16)}$$
$$1 \leqslant i, j \leqslant N/2-1$$

(note that $b_{2i,2j} = 0$ if $i > j$).

Thus the algorithm proceeds as follows:

(1) Compute the matrices α_{2i}, $1 \leqslant \frac{1}{2}N-1$, using (A.5).

(2) Compute the matrices γ_{2i}, $1 \leqslant i \leqslant \frac{1}{2}N-2$, using (A.9).

(3) Compute the upper triangular block-matrix $b_{2i,2j}$, using (A.14).

(4) Compute $(M^{-1})_{2i,2j}$, $0 \leqslant i, j \leqslant \frac{1}{2}N-1$, using the recursion relations (A.15)–(A.16). It should be noted that $b_{2i,2j}$ and $M_{2i,2j}$ can be held in one computer storage.

ACKNOWLEDGMENTS

The computations have been supported by the Centre de Calcul de l'Université Paris-Sud, Orsay. The authors express their gratitude to the referees for several valuable criticisms and suggestions. Helpful discussion with Professor G. Labrosse is also gratefully acknowledged.

REFERENCES

1. D. B. Haidvogel and T. Zang, *J. Comput. Phys.* **30**, 167 (1979).
2. P. Haldenwang, G. Labrosse, S. Abboudi, and M. Deville, *J. Comput. Phys.* **55**, 115 (1984).
3. D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM Monograph No. 26 (SIAM, Philadelphia, PA. 1977).
4. C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. Zang, *Spectral Methods in Fluid Dynamics* (Springer-Verlag, Berlin, 1977).
5. S. A. Orszag, *J. Fluid Mech.* **50**, 689 (1971).
6. C. Lanczos, *Applied Analysis* (Prentice–Hall, Englewood Cliffs, NJ, 1964).
7. J. W. Cooley and J. W. Tukey, *Math. Comput.* **19**, 297 (1965).
8. M. Deville and G. Labrosse, *J. Comput. Appl. Math.* **8**, 293 (1982).
9. C. W. Clenshaw, *Math. Comput.* **9**, 118 (1955).
10. N. Bressan and A. Quarteroni, *SIAM J. Numer. Anal.* **23**, 1138 (1986).
11. S. A. Orszag, *J. Comput. Phys.* **37**, 70 (1980).
12. D. Funaro, A. Quarteroni, and P. Zanolli, *SIAM J. Numer. Anal.* **25**, 1213 (1988).